Phase Direct CP Violations and General Mixing Matrices

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I formulate expressions for amplitudes suitable for quantifying both modulus and phase direct CP violations. They result in Möbius transformation (MT) relations, which provide encouraging information for the search of direct CP violations in general. I apply the formulation to calculate the measurements of phase direct CP violations and strong amplitudes in $B^{\mp} \to K^{\mp}\pi^{\pm}\pi^{\mp}$ by the Belle collaboration. For the formulation, I show a versatile construction procedure for $N \times N$ Cabibbo-Kobayashi-Maskawa (CKM) matrices, Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrices, and general unitary matrices. It clarifies the 3×3 cases and is useful for the beyond.

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Introduction — CP violation studies and observations have had a long interesting history [1-11]. CP violation in $B^{\mp}$ published in [1] was the first of its kind, direct and without particle-antiparticle oscillation. Further, in multiparticle decays, the total amplitudes, $A_{(tot)}$ and $\bar{A}_{(tot)}$, are coherent sums of amplitudes for various final resonances and backgrounds, $A_{(k)}$ and $\bar{A}_{(k)}$,

$$A_{(tot)} = \sum_{k=1}^{n} f_k A_{(k)} \quad \text{and} \quad \bar{A}_{(tot)} = \sum_{k=1}^{n} f_k \bar{A}_{(k)} \ , \quad (1)$$

where f_k are functions of invariant masses of some final particles. So phases of amplitudes can be measured, [1,2].

Here I derive general formulations capable of fully describe the phenomena and apply them to results of [1].

Expression A — Amplitudes, being complex valued, can always be expressed as

$$A = |A| e^{i\,\phi} \quad \text{and} \quad \bar{A} = |\bar{A}| e^{i\,\bar{\phi}}. \quad (2)$$

Direct CP violations are usually quantified by $\Delta_{cp} \neq 0$,

$$\Delta_{cp} \equiv (|A|^2 - |\bar{A}|^2)/(|A|^2 + |\bar{A}|^2), \quad (3)$$

where A and $\bar{A}$ represent $A_{(tot)}$ and $\bar{A}_{(tot)}$ or $A_{(k)}$ and $\bar{A}_{(k)}$ in Eqs.(1). (The symbol Δ_{cp} is used, in stead of $\mathcal{A}_{cp}$, to avoid confusion with amplitudes.)

Δ_{cp} is insensitive to the phase of $\bar{A}/A$, which is convention dependent. To describe phase CP violation we should use amplitudes, denoted by $\bar{A}'$ and A', which have the phase convention such that CP invariant amplitudes satisfy $\bar{A}'_{inv}/A'_{inv} = 1e^{i0}$. Then

$$\bar{A}'/A' = R_{cp}\; e^{i\Phi_{cp}} \equiv Z_{cp}, \quad -\pi < \Phi_{cp} \leq \pi, \quad (4)$$

and their deviations from $Z_{cp,inv} = 1e^{i0}$ give full quantifications of direct CP violations, modulus and phase.

Expression B — Belle [1] used the expressions for B^- and B^+ respectively (without stating model motivation),

$$A' = a e^{i\delta_B}(1 - b e^{i\varphi}), \;\; \bar{A}' = a e^{i\delta_B}(1 + b e^{i\varphi}), \quad (5)$$

$$\Delta_{cp} = -2b\cos\varphi / \left(1 + b^2\right). \quad (6)$$

I denote $B_{cp} \equiv b e^{i\varphi}$ and obtain MT conformal relations

$$\bar{A}'/A' \equiv Z_{cp} = (1 + B_{cp})/(1 - B_{cp}), \quad (7)$$

$$B_{cp} = -(1 - Z_{cp})/(1 + Z_{cp}). \quad (8)$$

Of the one-to-one and onto properties (circles/lines $\Leftrightarrow$ circles/lines) of MT, I point out some highlights. $b = 0 \Leftrightarrow Z_{cp} = 1e^{i0}$. So $b \neq 0$ gives CP violation, in modulus and phase allocated by b and φ. $[0 < b < 1,\ \varphi =^{\pi}_{0}] \Leftrightarrow \Phi_{cp} = 0$, thus all in $\Delta_{cp} = \pm 2b/\left(1 + b^2\right)$; $[b \neq 0,\ \varphi = \pm\pi/2] \Leftrightarrow \Delta_{cp} = 0$ ($R_{cp} = 1$), thus all in $\Phi_{cp} = \pm 2\arctan b$; maximal $\Phi_{cp} = \pi$ at $[1 < b,\ \varphi =^{\pi}_{0}]$; maximal $\Delta_{cp} = \pm 1$ at $[b = 1,\ \varphi =^{\pi}_{0}]$, where $[R_{cp} =^{0}_{\infty},\ \Phi_{cp}$ arbitrary].

Belle [1] assumed the nonresonant parts to be CP invariant and measured all $be^{i\varphi}$ and δ_B. I calculate $Z_{cp} = R_{cp}\; e^{i\Phi_{cp}}$ (versus only $\Delta_{cp} = \mathcal{A}_{cp}$ calculated in [1]), thus revealing their measurements of direct CP violations both in the modulus and in the phase, shown at the end with other quantities I derive after giving the realization of Expression B in the KM framework [7].

Direct CP violation in the KM framework — Direct CP violations come about naturally in the KM framework, as first established theoretically in K mesons (the s particles) [8]. Many other references and discussions can be found in the reviews of Particle Data Group (PDG) [12-14]. Here I give a self-contained discussion.

Weak decay amplitudes without particle-antiparticle oscillation are expressed as

$$A = V_{ub}V_{us}^{*}A_1 + V_{tb}V_{ts}^{*}A_2 \;\equiv z_1\; A_1 + z_2\; A_2, \quad (9)$$

$$\bar{A} = V_{ub}^{*}V_{us}A_1 + V_{tb}^{*}V_{ts}A_2 \equiv z_1^{*}\; A_1 + z_2^{*}\; A_2\ , \quad (10)$$

first for the $b \to s$ decays and then for decays with z_1 and z_2 as elements from the CKM matrix $\mathbb{V}$, [4,7].

One of the attributes of $\mathbb{V}$ is unitarity:

$$\sum_{m=1}^{3} V_{u_m d_j} V_{u_m d_k}^{*} = \delta_{jk}, \quad j,k = 1,2,3, \quad (11)$$

$$\sum_{j=1}^{3} V_{u_m d_j} V_{u_n d_j}^{*} = \delta_{mn}, \quad m,n = 1,2,3, \quad (12)$$

where the letters (u_m, d_j) denote the weak isospin doublets: (u,d), (c,s), (t,b) quarks [or (ν_e, e), (ν_μ, μ), (ν_τ, τ) leptons involving the PMNS matrix, [15].]

Eqs.(9,10) can be derived by drawing quark diagrams and combing terms with the same z. Because of $\sum_{l=1}^{3} z_l = 0$ conditions in Eqs.(11,12), A and $\bar{A}$ can always be expressed by two terms as in Eqs.(9,10) and in different ways. The strong amplitudes A_1, A_2 contain strong interactions to all orders. The relative phase of particle and antiparticle states is chosen such that the same strong amplitudes A_1, A_2 appear in A, $\bar{A}$. I will show that a suitable choice $\mathbb{V}'$ can be made to obtain A', $\bar{A}'$ (related to A, $\bar{A}$ by a phase transformation) so that phase direct CP violations can be quantified. That will be Expression C.

In [16], I did a comprehensive study of direct CP violation for c, b, and s particle decays. Writing

$$\Delta_{cp} = \frac{-4\,\mathrm{Im}\,(z_1^* z_2)\,\mathrm{Im}\,(A_1^* A_2)}{|z_1\ A_1 + z_2\ A_2|^2 + |z_1^*\ A_1 + z_2^*\ A_2|^2}, \qquad (13)$$

I found that all $\mathrm{Im}\,(z_1^* z_2) = \pm c_1 c_2 c_3\,(s_1)^2\, s_2 s_3 s_\delta$, in the notation of [7]. (This was four years before [17], whose parametrization has been called by PDG [12] the standard parametrization for the CKM and the PMNS matrices, thanks to the "advocation" and use by [14, 15].) The unique and ubiquitous $|\,\mathrm{Im}\,(z_1^* z_2)\,|$ found in Δ_{cp} were denoted by the symbol

$$X_{cp} \equiv |\,\mathrm{Im}\,(z_1^* z_2)\,| = c_{12}\,(c_{13})^2\, c_{23} s_{12} s_{13} s_{23} s_{\alpha_{13}}, \qquad (14)$$

in [17,18]. I have been using it since. It touches upon aspects and developments of the theory complimentary to those the symbol J does, [12-15]. It serves as a reminder that its relevance to experiments is through its role in direct CP violations.

Dividing the numerator and the denominator in Eq.(13) by $|z_1||z_2||A_1||A_2|$ (assuming none of them are zero for now) and simplifying, we obtain

$$\Delta_{cp} = -2\sin\theta\sin\Theta/(l + l^{-1} + 2\cos\theta\cos\Theta), \qquad (15)$$

where

$$\begin{aligned} \sin\theta &\equiv \mathrm{Im}(z_1^* z_2)/|z_1 z_2|, \quad r = |z_2|/|z_1|, \\ \sin\Theta &= \mathrm{Im}\,(A_1^* A_2)\,/|A_1 A_2|, \quad R = |A_2|/|A_1|; \text{ or} \\ re^{i\theta} &= z_2/z_1 \equiv z_{21}, \quad R\ e^{i\Theta} = A_2/A_1 \equiv A_{21}; \end{aligned} \qquad (16)$$

and $l \equiv rR$. The various $|\sin\theta|$ are

$$\sin\alpha_{d_j d_k} = X_{cp}/|(V_{ud_j} V_{ud_k}^*)(V_{td_j} V_{td_k}^*)|, \qquad (17)$$
$$\sin\beta_{d_j d_k} = X_{cp}/|(V_{td_j} V_{td_k}^*)(V_{cd_j} V_{cd_k}^*)|, \qquad (18)$$
$$\sin\gamma_{d_j d_k} = X_{cp}/|(V_{cd_j} V_{cd_k}^*)(V_{ud_j} V_{ud_k}^*)|, \qquad (19)$$

and similarly for $\sin\alpha_{u_m u_n}$, $\sin\beta_{u_m u_n}$, and $\sin\gamma_{u_m u_n}$. (In the case of $d_j d_k$ being bd , the α, β, γ notations conform to those in [12].) Each set of α, β, γ with the subscripts $d_j d_k$ (or $u_m u_n$) is associated with the $d_j d_k$ (or $u_m u_n$) orthogonal relation of Eqs.(11,12), thus the $d_j d_k$ (or $u_m u_n$) triangle on the complex plane. To get the signs of various $\sin\theta$, it is best to use a specific parametrization, like the standard parametrization or its variations (which are needed for reasons to be discussed). Amplitudes of a particular set of decays, Eqs.(9,10), involve one particular triangle; yet, once CP violation is established in one decay (as has been) all $|z| \neq 0$ and all $\sin\theta \neq 0$.

Variations to the standard parametrization in the standard construction — To define Z_{cp} and realize Expression B in the KM framework, I first show that the z_1 for a chosen A_1 can be made real and positive by using a suitable parametrization. [Note that all $A_{(k)}$and $\bar{A}_{(k)}$ in Eq.(1) can be made to have the same z_1].

In [17], besides the standard parametrization of $\mathbb{V}$, Keung and I found (by trials) a construction procedure for it: $\mathbb{V} = \mathbb{R}\,(23)\,\mathbb{U}\,(13)\,\mathbb{R}\,(12)$, one factor for each independent plane. $\mathbb{R}\,(jk)$ is the rotation matrix in the jk-plane and $\mathbb{U}\,(jk)$ is $\mathbb{R}\,(jk)$ with $\pm s_{jk} \to \pm s_{jk} e^{\mp i\alpha_{jk}}$,

$$\mathbb{U}\,(13) \equiv \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\alpha_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\alpha_{13}} & 0 & c_{13} \end{pmatrix}. \qquad (20)$$

(Symbols α_{jk} are used, saving δ_{jk} for the Kronecker deltas.) So the standard parametrization is $R\,(23)\,U\,(13)\,R\,(12)$-parametrization. The procedure also provides variations: $\mathbb{V}' = \mathbb{U}\,(23)\,\mathbb{R}\,(13)\,\mathbb{R}\,(12)$, or $\mathbb{V}'' = \mathbb{R}\,(23)\,\mathbb{R}\,(13)\,\mathbb{U}\,(12)$, or $\mathbb{V}s$ with different ordering of $(23)\,(13)\,(12)$. For $b \to s$, $b \to d$, and $s \to d$ decays, $\mathbb{V}'$ gives real positive $z_1' = V_{ub}' V_{us}'^*$, $V_{ub}' V_{ud}'^*$, or $V_{ud}' V_{us}'^*$.

Here I digress to give a fuller explanation of the above and formulate (what I would "advocate" to call) the *standard constructions* for $N \times N$ CKM, PMNS, and general unitary matrices. Let us start with the following.

The Murnaghan construction of $N \times N$ general unitary matrices [19]:

$$\mathbb{U} = \mathbb{F}\mathbb{A}, \quad \text{where} \quad \mathbb{A} \equiv \prod_{j<k\leq N} \mathbb{U}\,(jk)\,, \qquad (21)$$

$\mathbb{F}$= diag$(e^{i\phi_1}, e^{i\phi_2}, \cdots, e^{i\phi_N})$ and the $\frac{1}{2}N\,(N-1)$ number (one for each plane in N dimensions) of $\mathbb{U}\,(jk)$ are defined above Eq.(20). Different orderings of $\mathbb{U}\,(jk)$ give different (equally valid) parametrizations of $\mathbb{U}$.

$\mathbb{U}$ given by Eqs.(21) has all the attributes of a $N \times N$ unitary matrix. For example, there are $\frac{1}{2}N\,(N-1)$ angles and $[\frac{1}{2}N\,(N-1) + N]$= $\frac{1}{2}N\,(N+1)$ phases.

Theorem 1: The matrix $\mathbb{C}$ (let us call it the *core matrix*), transformed from $\mathbb{A}$ by the phase-stripping SPT:

$$\mathbb{C} = \mathbb{D}\mathbb{A}\mathbb{D}^\dagger = \prod_{j<k\leq(N-1),\ m<N} \mathbb{U}'\,(jk)\,\mathbb{R}\,(mN) \qquad (22)$$

with

$$\mathbb{D} = diag(e^{i\alpha_{1N}}, e^{i\alpha_{2N}}, \cdots, e^{i\alpha_{(N-1)N}}, 1), \qquad (23)$$

has the least possible number of phases under SPTs: $[\frac{1}{2}N\,(N-1) - (N-1)]$= $\frac{1}{2}\,(N-1)\,(N-2)$.

To prove the theorem, stick $\mathbb{D}^\dagger\mathbb{D}$ in-between all $\mathbb{U}\,(jk)$ of $\mathbb{A}$ in Eq.(21) and see that $\mathbb{D}\mathbb{U}\,(jk)\,\mathbb{D}^\dagger = \mathbb{U}'\,(jk)$ with $\alpha_{jk}' = \alpha_{jk} - \alpha_{jN} + \alpha_{kN}$, giving $\alpha_{jN}' = 0$ and $\mathbb{U}'\,(jN) = \mathbb{R}\,(jN)$. So the SPT maxes out the phase stripping from $\mathbb{A}$ [and all $\mathbb{A}$ with different ordering of $\mathbb{U}\,(jk)$]. **Corollary 1**: The phases in a core matrix $\mathbb{C}$ can be moved around by phase-moving SPT (see an example of it later).

Using $\mathbb{C}$ we can make the following explicit constructions. Let us call them the standard constructions.

The standard construction of $N\times N$ general unitary matrices [revealing more phase structures than the Murnaghan construction, Eq.(21)]:

$$\mathbb{U}=\mathbb{F}\mathbb{D}^{\dagger}\mathbb{C}\mathbb{D}\equiv\mathbb{F}^{'}\mathbb{C}\mathbb{D}. \tag{24}$$

The standard construction of $N\times N$ CKM matrices for quarks:

$$\mathbb{V}=\mathbb{C}^{q}. \tag{25}$$

Theorem 1 and Eq.(24) show by explicit construction how the usual phase counting works out. $(2N-1)$ out of the $2N$ phase freedoms of quark fields are used to strip away all that can be from the up-down quark mixing matrix $\mathbb{U}^q$ by phase transformations (PT): $\mathbb{C}^q=\mathbb{F}^{\prime\dagger}\mathbb{U}^q\mathbb{D}^{\dagger}$, Eq.(24) to Eq.(25). So, always one phase is left free. It can be used to make one (only one) of the many A_1 in Eq.(1) real. In K decays, setting zero-isospin-change amplitude real is the Wu-Yang phase convention [6].

What Keung and I found by trials in [17] is the 3×3 forerunner of this standard construction and [18] extended it to a 4×4 case. An example of Corollary 1 is the phase-moving SPT, $\mathbb{V}'=\mathrm{diag}(1,1,e^{-i\alpha_{13}})\mathbb{V}\mathrm{diag}(1,1,e^{i\alpha_{13}})$ and $\alpha'_{23}=-\alpha_{13}$. Other such relations among $\mathbb{V}$, $\mathbb{V}'$, $\mathbb{V}''$ are left as exercises.

The standard construction of $N\times N$ PMNS matrices for Dirac leptons and for Majorana ν:

$$\mathbb{V}^{l_D}=\mathbb{C}^{l_D},\quad \text{and}\quad \mathbb{V}^{\nu_M}=\mathbb{C}^{\nu_{\mathbb{M}}}\mathbb{D}. \tag{26}$$

Dirac lepton fields have the same phase freedoms as quarks fields, so $\mathbb{V}^{l_D}$ is given by a core matrix as is $\mathbb{V}$ for quarks. However, Majorana neutrino fields do not have phase freedom [15], so only the phase freedoms of the Dirac leptons can be used to strip away N phases from the Dirac-Majorana mixing matrix $\mathbb{U}^{\nu_{\mathbb{M}}}$ by one PT: $\mathbb{C}^{\nu_{\mathbb{M}}}\mathbb{D}=\mathbb{F}^{\prime\dagger}\mathbb{U}^{\nu_{\mathbb{M}}}$, Eq.(24) to Eq.(26). What has been adapted in neutrino research [15] is the 3×3 case of [17] for $\mathbb{C}^{\nu_{\mathbb{M}}}$. All variations discussed here can also apply.

In Eq.(21) I can also use $\mathbb{U}=\mathbb{A}'\mathbb{F}$ with $\mathbb{A}'\equiv\mathbb{F}\mathbb{A}\mathbb{F}^{\dagger}$, follow similar procedure and obtain another core matrix $\mathbb{C}'=\mathbb{D}'\mathbb{A}'\mathbb{D}'^{\dagger}$ for $\mathbb{U}$. I also have theorems that give different constructions with core matrices involving less than $\frac{1}{2}N\,(N-1)$ planes, like the Euler construction for $SO\,(3)$. However, I see no advantage over the standard construction for the uses discussed here. Further, I can use the core matrices to give spectral constructions for matrices. I give details of these results in [20].

Expression C — Representing CP invariant amplitudes by $A_1\neq 0$ and using the $\mathbb{V}'$ in which $z'_1=|z_1|$, I obtain Expression C:

$$A'=|z_1|A_1(1+z_{21}A_{21})=de^{i\delta_1}(1+le^{i(\Theta+\theta)}), \tag{27}$$

$$\bar{A}'=|z_1|A_1(1+z^*_{21}A_{21})=de^{i\delta_1}(1+le^{i(\Theta-\theta)}), \tag{28}$$

where $de^{i\delta_1}\equiv|z_1|A_1$ and relations given by Eqs.(16) still hold – good exercise to check; and

$$\bar{A}'/A'\equiv Z_{cp}=(1+z^*_{21}A_{21})/(1+z_{21}A_{21}), \tag{29}$$

$$A_{21}=(1-Z_{cp})/(z_{21}-z^*_{21}Z_{cp}). \tag{30}$$

Now the CP invariant $\bar{A}'_{inv}/A'_{inv}=1e^{i0}$. What we have done above is equivalent (and gives justification) to starting with $\mathbb{V}$ and making the phase change to amplitudes: $A'=e^{-i\,\alpha_1}A$ and $\bar{A}'=e^{i\,\alpha_1}\bar{A}$ with $e^{i\,\alpha_1}\equiv z_1/|z_1|$.

Z_{cp} is in terms of the knowables (measurable in principle), z_{21} and A_{21}, and is TM conformally related to A_{21} (not z_{21}). Their one-to-one and onto mapping properties carry important information. $\mathrm{Im}z_{21}|A_{21}|=l\sin\theta=0\Leftrightarrow Z_{cp}=1e^{i0}$. So $l\sin\theta\neq 0$ assures CP violation, in modulus or phase allocated by $l\sin\theta$ and Θ. Δ_{cp} is still given by Eq.(15). Since all $|z|\neq 0$ and all $\sin\theta\neq 0$ (i.e., $\mathrm{Im}z_{21}=r\sin\theta\neq 0$ and well defined), so in all decays direct CP violations happen everywhere on the whole A_{21} complex plane except one point, $A_{21}=0$. [For example: at $\Theta=0$, $\Delta_{cp}=0$ and $\Phi_{cp}=-2\arctan[l\sin\theta/(1+l\cos\theta)]$; and at $\Theta=\pi\pm\theta$ and $l=1$, $\Delta_{cp}=\pm 1$.] Further, for any value of $\mathrm{Im}z_{21}=r\sin\theta\neq 0$, CP violation can be large if A_{21} cooperates. Eq.(30) gives the A_{21} for a Z_{cp} asked for.

Besides giving the conceptual understanding mentioned above and the realization of Expression B to be discussed below, Expression C also gives the possibility of finding z_{21}, A_{21} and $|z_1|A_1$ from data via Eqs.(27,28); and provides versatile ways of analyzing data. [If the data are not sensitive to these many parameters, one can put in z_{21} from [12] and find A_{21} and $|z_1|A_1$.]

Realization of Expression B in the KM framework and derivation of amplitudes from Belle [1] — Using Eqs.(27,28) of Expression C, decomposing $e^{\pm i\theta}$ into real and imaginary parts, identifying

$$ae^{i\delta_B}=|z_1|A_1(1+A_{21}r\cos\theta), \tag{31}$$

$$B_{cp}=-iA_{21}r\sin\theta/\left(1+A_{21}r\cos\theta\right), \tag{32}$$

I obtain the realization of Expression B, Eqs.(5), in terms of knowables in the KM framework.

Besides $\bar{A}'/A'\equiv Z_{cp}=R_{cp}\,e^{i\Phi_{cp}}$ mentioned earlier, Eqs.(7), I now can derive from Eqs.(31, 32)

$$A_{21}=-B_{cp}/[ir\sin\theta+B_{cp}r\cos\theta]\quad\text{and} \tag{33}$$

$$A_1=ae^{i\delta_B}/[|z_1|(1+A_{21}r\cos\theta)] \tag{34}$$

in terms of B_{cp}. The MT conformal relations between B_{cp} and A_{21}, Eqs.(32,33), are anticipated from the MT relations that Z_{cp} has both with B_{cp}, Eqs.(7,8), and with A_{21}, Eqs(29,30).

Substituting into Eqs.(31,32) δ_B and $B_{cp}\equiv be^{i\varphi}$ from [1], and $\sin\theta=-\sin\alpha_{bs}\approx -0.82$, $\cos\theta\approx -0.57$, and $r=|V_{tb}V^*_{ts}|/|V_{ub}V^*_{us}|\approx 46$ (derived using [12,14] and assuming all angles in the standard parametrization of

$\mathbb{V}$ being in the first quadrant), I obtain the values of $A_{21} \equiv A_2/A_1$ and $\tilde{A}_1 \equiv (|z_1|/a)A_1$, listed below together with $Z_{cp} = R_{cp}\, e^{i\Phi_{cp}}$, for $B^{\mp}$ decays into {1} $K^*(892)\,\pi^{\mp}$, {2} $K^*(1430)\,\pi^{\mp}$, {3} $\rho^0(770)\,K^{\mp}$, {4} $f_2(1270)\,K^{\mp}$, the best four from [1]:

{1} $Z_{cp} = 1.16\exp(\text{-}i0.048)$,
$A_{21} = 0.0021\exp(\text{-}i1.8)$, $\tilde{A}_1 = 0.98\exp(\text{-}i0.052)$;

{2} $Z_{cp} = 0.93\exp(\text{-}i0.12)$,
$A_{21} = 0.0019\exp(i2.5)$, $\tilde{A}_1 = 0.96\exp(i0.99)$;

{3} $Z_{cp} = 0.74\exp(\text{-}i0.46)$,
$A_{21} = 0.0087\exp(i2.4)$, $\tilde{A}_1 = 0.85\exp(\text{-}i0.24)$;

{4} $Z_{cp} = 1.98\exp(\text{-}i0.34)$,
$A_{21} = 0.011\exp(\text{-}i1.7)$, $\tilde{A}_1 = 0.93\exp(i2.2)$.

Only central values are shown. The proper way to find errors in R_{cp} and Φ_{cp} is to analyze data distributions in Z_{cp} by authors of Belle [1]. However, I did carry out various error calculations using statistical errors in b and φ given by [1] and noticed the following. When an error in φ decreases (increases) modulus CP violation, it increases (decreases) phase CP violation; in contrast, when an error decreases (increases) b, both the modulus and the phase CP violations decrease (increase). The phase CP violation, $\Phi_{cp} \neq 0$, in case {2} stood out, [21].

$A_1 = (a/|z_1|)\tilde{A}_1$ can be derived once Belle publishes values of a, using partial rates and f_k of Eq.(1). (Note the wide range of the central values of the moduli and phases of $\tilde{A}_1$ and A_{21}. The proper way to obtain them and their error analyses will be to fit data using Expression C, [21].) These A_1 and A_2 from experiments can be compared with theory. (For current theoretical calculation schemes, see [22,23], e.g.). Alternatively, use A_1 and A_2 from theory in Eqs.(31,32), then solve for r and θ, and compare them with those obtained elsewhere.

Conclusion — The formulations given here have general applications for studying phase and modulus direct CP violations and strong amplitudes in weak decays, beyond the results calculated here for $B^{\mp} \to K^{\mp}\pi^{\pm}\pi^{\mp}$ of [1]. The Möbius (linear fractional conformal) transformation relations found here tell us that in the KM formulation for CP violation, once it is established in one reaction (as has been), the amount of direct CP violations (phase and modulus) in all decays are unrestricted by the CKM matrix, but solely dependent on how cooperative the strong amplitudes are. This new understanding is encouraging for the search of direct CP violations in general. The versatile procedure given here for the constructions of $N \times N$ CKM, PMNS, and general unitary matrices clarifies the 3×3 cases and is useful for the beyond.

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[21] I look forward to discussing with the authors of [1] about analyzing their data in Z_{cp} (so to find the errors to the central values of phase direct CP violations, $\Phi_{cp} \neq 0$, calculated here) and in Expression C (so to find the physical quantities in the KM framework and their error analyses in one fell swoop, as discussed here).

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